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ANALYSIS OF DISSIMILAR ANISOTROPIC WEDGES SUBJECTED TO ANTIPLANE SHEAR DEFORMATION

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Abstract—The antiplane strain problem of two dissimilar anisotropic composite wedges of arbitrary angles that are bonded together along a common edge is considered. The surfaces of the wedges can be subjected to traction–traction, traction–displacement or displacement–displacement boundary conditions. The dependence of the order of the stress singularity on the wedge angles and material constants is studied. The angular distribution of stresses at the apex and the exact full field stress solutions are also investigated. Explicit solutions for the order of stress singularity are obtained for some special cases. It is found that the order of the stress singularity is always real for the antiplane dissimilar anisotropic wedge problem. This is quite different from the in-plane case, in which the complex type of stress singularity might exist.

1. INTRODUCTION

The problem of finding the stress singularities at the apex of an isotropic elastic wedge was first considered by Williams (1952) by using the eigenfunction-expansion method. Williams found that the stresses near the apex are proportional to $r^{-\lambda}$ and the value of λ can be real or complex in general. Tranter (1948) used the Mellin transform in conjunction with the Airy stress function representation of plane elasticity to solve for the isotropic wedge problem. Williams (1959) obtained the solution of dissimilar materials with a semi-infinite crack. It was discovered for the first time that the stresses possess a sharp oscillatory character (i.e. complex λ). Bogy (1971) used the Mellin transform to treat the problem of two materially dissimilar isotropic elastic wedges of arbitrary angles that are bonded together along a common edge and subjected to surface traction at the boundary. A number of other workers have studied similar problems (see Dempsey and Sinclair, 1981 and Erdogan and Gupta, 1972, for example).

Investigation of associated wedge problems for anisotropic materials started from Benthem (1963) and Sih *et al.* (1965). Following the approach of Stroh (1958, 1962), Ting and Chou (1981) and Ting (1986) studied the stress distribution near the composite wedge of anisotropic materials. Bogy (1972), Kuo and Bogy (1974a, 1974b) employed a complex function representation of the solution (Green and Zerna, 1954) in conjunction with a generalized Mellin transform to analyze stress singularities in an anisotropic wedge. Several studies in this area have been made in the last decade (see Clements, 1971; Delale and Erdogan, 1979; Hoenig, 1982; Wang and Choi, 1982a, b).

In this paper, antiplane strain problems of general anisotropic dissimilar elastic wedges of arbitrary angles that are assumed to be perfectly bonded together along a common edge are considered. Here the problem of traction (or displacement) prescribed on both wedge faces, and the problem of traction prescribed on one face with displacement prescribed on the other are solved. The open two-dimensional regions occupied by the cross-sections of two wedges of angles α and β, and their common boundary lying in the positive x-axis are shown in Fig. 1. The problem will be solved by a straightforward application of the Mellin transform in conjunction with the stress function, as performed by Tranter (1948) and Bogy (1971, 1972). We focus our attention on the dependence of the order of the stress singularity on the wedge angles, material constants and boundary conditions. The angular dependence of the stress field near the wedge and the full field stresses distribution are also analyzed. Unlike the existence of the oscillatory character (i.e. complex λ) for the singular behavior near the bimaterial wedge for the in-plane problem, we found that the order of stress singularity λ is real for a general anisotropic bimaterial wedge of the antiplane strain

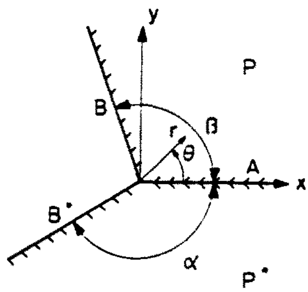

Fig. 1. Configuration of bonded dissimilar wedges.

problem subjected to different boundary conditions (i.e. traction–traction, displacement–displacement, traction–displacement). Furthermore, if an "effective angle" is introduced, then the order of the singularity for the general anisotropic material can be obtained easily from the solution of the isotropic case.

2. STATEMENT OF PROBLEM AND GENERAL SOLUTION IN MELLIN TRANSFORM DOMAIN

Let P, P^* denote the open two-dimensional regions occupied by two wedges of angles α, β $(\alpha+\beta \leqslant 2\pi)$ and A the straight segment of their boundaries in common as shown in Fig. 1. The remaining straight boundary segments are denoted by B and B^*. For the antiplane shear deformation, the only nonvanishing displacement component is along the z-axis, $w(x, y)$. In the absence of body forces, the equilibrium equation for a homogeneous isotropic material is given by

$$\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}=0. \tag{1}$$

The nonvanishing stresses are

$$\tau_{rz}=\mu\frac{\partial w}{\partial r} \tag{2}$$

$$\tau_{\theta z}=\frac{\mu}{r}\frac{\partial w}{\partial \theta}. \tag{3}$$

The boundary conditions considered on the straight boundary segments B, B^* could be traction–traction, displacement–displacement or traction–displacement. In addition, we shall require the stress fields to satisfy the regularity conditions

$$\tau_{rz}, \tau_{\theta z}=O(r^{-1+\delta}) \quad \text{as} \quad r\to\infty \quad \text{for} \quad \delta>0. \tag{4}$$

The Mellin transform method is a very convenient tool for solving the boundary-value problem and the form of this solution is particularly suitable for asymptotic analysis of the stress field at the wedge apex. Let the Mellin transform of a function $f(r)$ be denoted by $\hat{f}(s)$:

$$\hat{f}(s)=M\{f;s\}=\int_0^\infty f(r)r^{s-1}\,\mathrm{d}r \tag{5}$$

where s is a complex transform parameter. The Mellin transforms of $w(r,\theta)$, $r\tau_{rz}(r,\theta)$, $r\tau_{\theta z}(r,\theta)$ with respect to r are denoted by $\hat{w}(s,\theta)$, $\hat{\tau}_{rz}(s,\theta)$ and $\hat{\tau}_{\theta z}(s,\theta)$. Thus

$$\hat{w}(s,\theta) = \int_0^\infty w(r,\theta) r^{s-1}\, \mathrm{d}r \tag{6}$$

$$\hat{\tau}_{rz}(s,\theta) = \int_0^\infty \tau_{rz}(r,\theta) r^{s}\, \mathrm{d}r \tag{7}$$

$$\hat{\tau}_{\theta z}(s,\theta) = \int_0^\infty \tau_{\theta z}(r,\theta) r^{s}\, \mathrm{d}r. \tag{8}$$

By use of the inversion theorem for the Mellin transform, the stresses and displacement components are given by

$$w(r,\theta) = \frac{1}{2\pi \mathrm{i}} \int_{\rho-\mathrm{i}\infty}^{\rho+\mathrm{i}\infty} \hat{w}(s,\theta) r^{-s}\, \mathrm{d}s \tag{9}$$

$$\tau_{rz}(r,\theta) = \frac{1}{2\pi \mathrm{i}} \int_{\rho-\mathrm{i}\infty}^{\rho+\mathrm{i}\infty} \hat{\tau}_{rz}(s,\theta) r^{-s-1}\, \mathrm{d}s \tag{10}$$

$$\tau_{\theta z}(r,\theta) = \frac{1}{2\pi \mathrm{i}} \int_{\rho-\mathrm{i}\infty}^{\rho+\mathrm{i}\infty} \hat{\tau}_{\theta z}(s,\theta) r^{-s-1}\, \mathrm{d}s. \tag{11}$$

Because of condition (4), the path of integration in the complex line integrals Re $(s) = \rho$ in (9), (10) and (11) must lie within a common strip of regularity of their integrands, the choice of ρ is taken to be

$$\rho = -\varepsilon \quad 0 < \varepsilon < (|\mathrm{Re}\,(s_1)|) \tag{12}$$

where s_1 denotes the location of the pole in the open strip $-1 < \mathrm{Re}\,(s) < 0$ with the largest real part and Re denotes the real part of the complex argument.

Applying the Mellin transform (6) to (1) yields an ordinary differential equation for $\hat{w}$, the general solution of which is

$$\hat{w}(s,\theta) = a(s)\sin(s\theta) + b(s)\cos(s\theta) \tag{13}$$

in which the functions $a(s)$ and $b(s)$ are to be determined through the transforms of the boundary and continuity conditions. The stress components in the transformed form appear as

$$\hat{\tau}_{rz}(s,\theta) = -\mu s \hat{w}(s,\theta) \tag{14}$$

$$\hat{\tau}_{\theta z}(s,\theta) = \mu \frac{\mathrm{d}\hat{w}(s,\theta)}{\mathrm{d}\theta}. \tag{15}$$

3. ASYMPTOTIC BEHAVIOR AND THE STRESS SINGULARITIES AT WEDGE APEX

We will discuss the three possible combinations of boundary conditions separately.

Case I. Traction–traction boundary condition

Perfect bonding along the interface $\theta = 0$ is ensured by the stress and displacement continuity conditions, and the traction boundary conditions on the other wedge segments are given as follows.

$$\tau^*_{\theta z}(r,-\alpha) = t^*(r), \qquad \tau_{\theta z}(r,\beta) = t(r),$$
$$\tau^*_{\theta z}(r,0) = \tau_{\theta z}(r,0), \quad w^*(r,0) = w(r,0). \tag{16}$$

In (16), $t^*(r)$ and $t(r)$ represent the shearing tractions prescribed on B^* and B. Substitution of (13)–(15) into the Mellin transform of (16) provides the following four equations for the four unknown functions $a^*(s)$, $b^*(s)$, $a(s)$, $b(s)$:

$$\begin{aligned}\mu^*a^*(s)\cos(\alpha s)+\mu^*b^*(s)\sin(\alpha s) &= \hat{t}^*(s)/s\\ \mu a(s)\cos(\beta s)-\mu b(s)\sin(\beta s) &= \hat{t}(s)/s\\ \mu^*a^*(s)-\mu a(s) &= 0\\ b^*(s)-b(s) &= 0\end{aligned} \tag{17}$$

where $\hat{t}^*(s)$, $\hat{t}(s)$ denote the Mellin transforms of $t^*(r)$ and $t(r)$. The solution of (17) together with (13)–(15) determine the exact solutions of stresses $\hat{\tau}^*_{rz}(s,\theta)$, $\hat{\tau}^*_{\theta z}(s,\theta)$, and $\hat{\tau}_{rz}(s,\theta)$, $\hat{\tau}_{\theta z}(s,\theta)$ in the transformed form on $-\alpha<\theta<0$ and $0<\theta<\beta$, respectively.

$$\hat{\tau}_{rz}(s,\theta) = -\frac{1}{D}\{[\mu\hat{t}^*(s)\sin(\beta s)+\mu^*\hat{t}(s)\sin(\alpha s)]\sin(s\theta) + [\mu\hat{t}^*(s)\cos(\beta s)-\mu\hat{t}(s)\cos(\alpha s)]\cos(s\theta)\} \tag{18}$$

$$\hat{\tau}_{\theta z}(s,\theta) = \frac{1}{D}\{[\mu\hat{t}^*(s)\sin(\beta s)+\mu^*\hat{t}(s)\sin(\alpha s)]\cos(s\theta) - [\mu\hat{t}^*(s)\cos(\beta s)-\mu\hat{t}(s)\cos(\alpha s)]\sin(s\theta)\} \tag{19}$$

in which

$$D(\alpha,\beta,\mu^*,\mu,s) = \mu^*\sin(\alpha s)\cos(\beta s)+\mu\sin(\beta s)\cos(\alpha s). \tag{20}$$

Expressions similar to those in (18) and (19) follow in the same manner for $\hat{\tau}^*_{rz}(s,\theta)$ and $\hat{\tau}^*_{\theta z}(s,\theta)$. From (18) and (19), it is clear that $\hat{\tau}_{rz}(s,\theta)$, $\hat{\tau}_{\theta z}(s,\theta)$ etc., are meromorphic functions of s for fixed θ in $-1<\mathrm{Re}(s)<0$ whose poles can occur only at the zeros of $D(s)$ in the open strip. We can now indicate the appropriate path of integration for the inversion integrals in (10) and (11). We may then choose the path of integration for the inversion integrals to lie within the common strip of regularity $\mathrm{Re}(s_1)<\rho<0$ with s_1 denoting the zero of $D(s)$ with the largest real part in the strip.

The largest contribution for the asymptotic behavior of the stress field as $r\to 0$ depends on the location of the largest real root s_1 of $D(s)$, and is given by

$$\begin{aligned}\lim_{r\to 0}\tau_{rz}(r,\theta) &= \lim_{s\to s_1} r^{-(s+1)}(s-s_1)\hat{\tau}_{rz}(s,\theta)+o(r^{-\lambda}), \quad 0<\theta<\beta\\ &= r^{-\lambda}\lim_{s\to s_1}\frac{A(s)}{D'(s)}[\cos(s\theta)-R\tan(s\alpha)\sin(s\theta)]\end{aligned} \tag{21}$$

$$\begin{aligned}\lim_{r\to 0}\tau^*_{rz}(r,\theta) &= \lim_{s\to s_1} r^{-(s+1)}(s-s_1)\hat{\tau}^*_{rz}(s,\theta)+o(r^{-\lambda}), \quad -\alpha<\theta<0\\ &= r^{-\lambda}\lim_{s\to s_1}\frac{A(s)}{D'(s)}R[\cos(s\theta)-\tan(s\alpha)\sin(s\theta)]\end{aligned} \tag{22}$$

where

$$A(s) = -\mu[f^*(s)\cos(s\beta) - f(s)\cos(s\alpha)]$$

and primes denote differentiation with respect to s. If s_1 is a simple zero of $D(s)$, then the type of singularity will be of the order $\lambda = \mathrm{Re}(s_1)+1$. Obviously if s_1 is a complex zero, then the stress fields are oscillatory in the limit $r \to 0$. If no zero of $D(s)$ occurs in $-1 < \mathrm{Re}(s) < 0$, but $\mathrm{d}D(s)/\mathrm{d}s = 0$ at $s = -1$, then it will have a logarithm type singularity. Hence, determining the location of the zeros of the function $D(s)$ in the strip $-1 < \mathrm{Re}(s) < 0$ is our principal task. It is shown in the Appendix that the zeros of $D(s)$ are always real for any combination of material constants and wedge angles, so that the possibility of the oscillatory singular behavior is precluded. The jump value of τ_{rz} near the interface as $r \to 0$ will be

$$\lim_{r\to 0}[\tau_{rz}(r,0^+) - \tau^*_{rz}(r,0^-)] = (1-R)r^{-(s_1+1)}\frac{A(s_1)}{D'(s_1)}.$$

Equation $D = 0$ in (20) can be rewritten as

$$\frac{\sin[(\alpha+\beta)s]}{\sin[(\alpha-\beta)s]} = \frac{1-R}{1+R} \tag{23}$$

where $R = \mu^*/\mu$ is the ratio of the shear moduli of the two materials. Here solutions for s depend on the single bimaterial parameter R and the wedge angles α, β. In order to recover some previously known results, we examine $D(s)$ for various limiting cases. If the material in P is infinitely rigid, then we must consider the limit $\mu \to \infty$ with μ^* constant. If the material in P^* is assumed to have no rigidity, then we must consider the limit $\mu^* \to 0$ with μ constant. The limit $R \to 0$ describes both of these two limiting cases and (23) becomes in this limit $\sin(\beta s)\cos(\alpha s) = 0$. Equating the first factor to zero, $\sin\beta s = 0$, we recover the solution of an elastic wedge of angle β subject to the traction–traction boundary condition. By setting the second factor to zero, $\cos\alpha s = 0$, the problem reduces to that of a wedge of angle α with one face rigidly clamped and the other face subjected to the traction boundary condition. Suppose that the shear moduli of the materials occupying P^* and P are the same, that is $R = 1$ and $\alpha+\beta = \gamma$. It is not difficult to verify that $D(s) = 0$ can be written as $\sin(\gamma s) = 0$. For the case of equal angle wedges, i.e. $\alpha = \beta$, we have $\sin 2\alpha s = 0$ or $\lambda = 1-\pi/2\alpha$, hence the order of stress singularity is independent of the two material constants μ and μ^*. When $\alpha = \beta = \pi$, the problem becomes that of two dissimilar materials with cracks or fault lines along their common interface and the familiar square root singularity is obtained.

We now turn to the numerical computation of the zeros of $D(s)$ as given in (23). The results of the numerical computations are given in Figs 2 and 3, which show the dependence

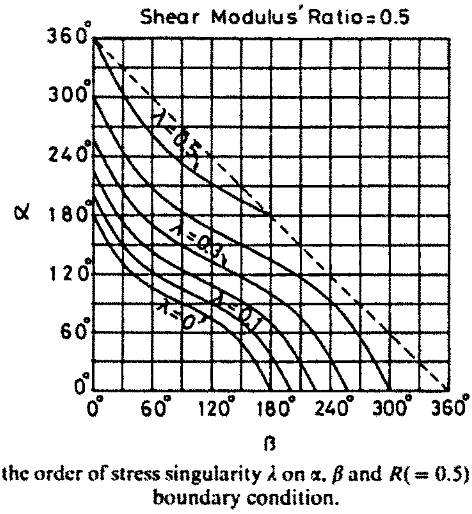

Fig. 2. Dependence of the order of stress singularity λ on α, β and $R(=0.5)$ for the traction–traction boundary condition.

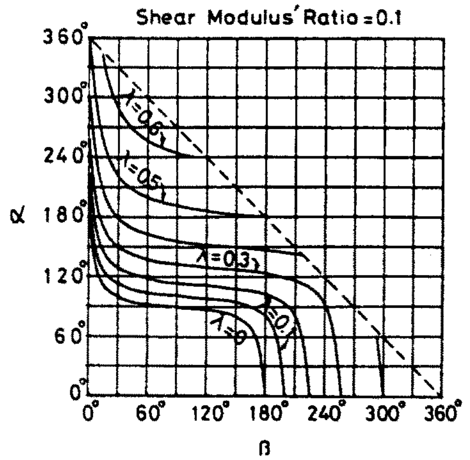

Fig. 3. Dependence of the order of stress singularity λ on α, β and $R(=0.1)$ for the traction–traction boundary condition.

of the order of stress singularity λ on $R = \mu^*/\mu$ $(= 0.5, 0.1)$ for various values of α, β. When curves corresponding to different values of λ overlap, i.e. when multiple roots occur in $0 < \lambda < 1$, it is understood that the larger value of λ is plotted which controlled the asymptotic stress as $r \to 0$. The angular dependence of stresses near the wedge apex for $\alpha = \pi$, $\beta = \pi/2$ and $R = 2$ as shown in (21) and (22) [only the angular dependent term is calculated, e.g. $\cos(s\theta) - R\tan(s\theta)\sin(s\theta)$ in (21)] is plotted in Fig. 4. The shear stress $\tau_{\theta z}$ is continuous on the bonded edge $\theta = 0$ while τ_{rz} is discontinuous at the interface.

The exact full field shear stresses of τ_{rz} and $\tau_{\theta z}$ are computed numerically for $\alpha = \pi$, $\beta = \pi/2$ and $R = 2$. The specific loading considered here is that of a uniform shear stress $\tau_{\theta z}$ with unit magnitude applied from $r = 0$ to $r = 1$ on the boundary B and B^*. Thus, the load functions on the boundary will be

$$t^*(r) = H(1-r), \quad t(r) = -H(1-r)$$

where H is the Heaviside function. The results of the computations of stresses along the bonded edge $\theta = 0$ and different angles $\theta = 30°$, $60°$ and the stresses near the wedge boundary $\theta = 75°$, $80°$, $85°$ are exhibited in Figs 5 and 6.

Case II. Traction–displacement boundary condition

Here the problem of traction prescribed on one face with displacement prescribed on the other is solved. The solutions presented follow the outline established previously. Thus we consider the following boundary conditions,

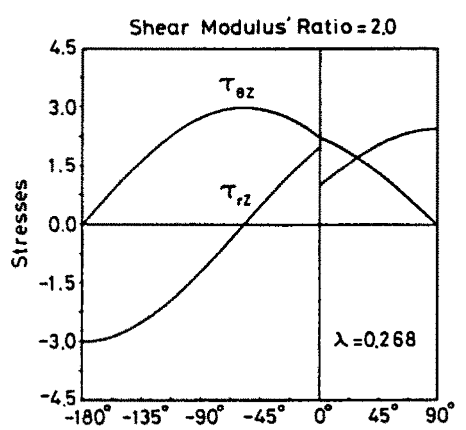

Fig. 4. The angular distribution of stresses $\tau_{\theta z}$ and τ_{rz} of the asymptotic behavior as $r \to 0$ for the traction-traction boundary condition ($\alpha = \pi$, $\beta = \pi/2$, $R = 2.0$).

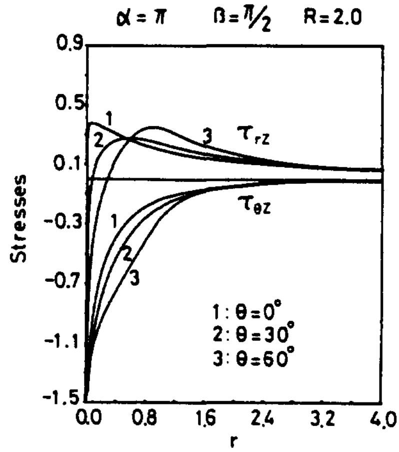

Fig. 5. Stresses $\tau_{\theta z}$ and τ_{rz} of dissimilar isotropic wedges with a half plane bonded to a quarter plane for $\theta = 0^\circ$, 30° and 60°.

$$w^*(r, -\alpha) = W^*(r)$$

$$\tau_{\theta z}(r, \beta) = t(r). \tag{24}$$

The solutions of stresses in the transformed form is given by

$$\hat{\tau}_{rz}(s, \theta) = -\frac{\mu^*}{D}\{[s\mu^*\hat{W}^* \sin(s\beta) + R\hat{t}(s)\cos(s\alpha)]\sin(s\theta) + [s\mu^*\hat{W}^*\cos(s\beta) + \hat{t}(s)\sin(s\alpha)]\cos(s\theta)\} \tag{25}$$

$$\hat{\tau}_{\theta z}(s, \theta) = \frac{\mu}{D}\{[s\mu\hat{W}^* \sin(s\beta) + \hat{t}(s)\cos(s\alpha)]\cos(s\theta) - [s\mu^*\hat{W}^*\cos(s\beta) + \hat{t}(s)\sin(s\alpha)]\sin(s\theta)\} \tag{26}$$

where

$$D(s) = \mu^*\cos(\alpha s)\cos(\beta s) - \mu\sin(\alpha s)\sin(\beta s). \tag{27}$$

The roots of (27) must satisfy the equation

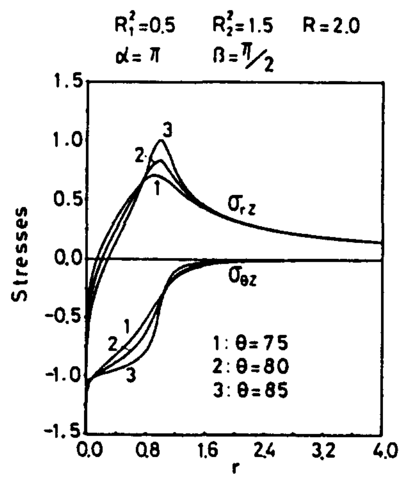

Fig. 6. Stresses $\tau_{\theta z}$ and τ_{rz} of dissimilar isotropic wedges with a half plane bonded to a quarter plane for $\theta = 75^\circ$, 80° and 85°.

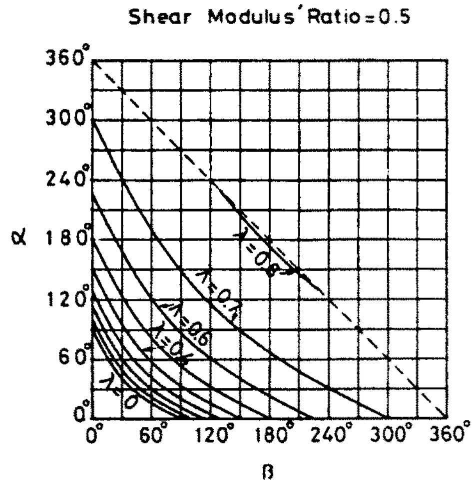

Fig. 7. Dependence of the order of stress singularity λ on α, β and $R(=0.5)$ for the traction–displacement boundary condition.

$$\frac{\cos[(\alpha+\beta)s]}{\cos[(\alpha-\beta)s]} = \frac{1-R}{1+R}. \tag{28}$$

Equation (28) does not change by interchanging α and β, hence the order of stress singularity λ as shown in Figs 7 and 8 is symmetric with respect to the line $\alpha = \beta$. For $\alpha+\beta = c$, then the case of equal angle $\alpha = \beta = c/2$ has the largest value of λ and hence the most severe stress singularity. For equal angle wedges, that is $\alpha = \beta$, we have $\cot^2 s\alpha = \mu/\mu^*$. Hence the order of singularity for this case is

$$\lambda = 1 - \frac{\pi}{2\alpha} + \frac{1}{\alpha}\tan^{-1}\left(\frac{\mu}{\mu^*}\right)^{1/2}.$$

For the crack geometry, $\alpha = \beta = \pi$, the order of stress singularity is in agreement with that obtained by Ting (1986), that is

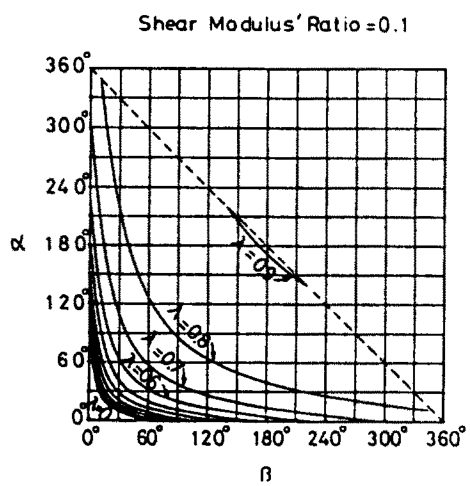

Fig. 8. Dependence of the order of stress singularity λ on α, β and $R(=0.1)$ for the traction–displacement boundary condition.

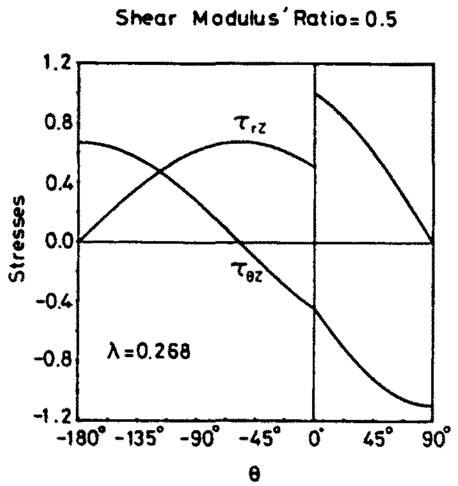

Fig. 9. The angular distribution of stresses $\tau_{\theta z}$ and τ_{rz} of the asymptotic behavior as $r \to 0$ for the traction–displacement boundary condition ($\alpha = \pi$, $\beta = \pi/2$, $R = 0.5$).

$$\lambda = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \left(\frac{\mu}{\mu^*}\right)^{1/2}. \tag{29}$$

The dependence of the order of singularity λ on the material property for the interfacial crack is shown in (29), if the two materials are the same ($\mu = \mu^*$), then we have the familiar value of λ equal to 3/4. The asymptotic behavior of the stress field is given by

$$\lim_{r\to 0} \tau_{rz} = \lim_{s\to s_1} r^{-(s+1)}(s-s_1)\hat{\tau}_{rz}(s,\theta) = r^{-\lambda} \lim_{s\to s_1} \frac{B(s)}{D'(s)}[\cos(s\theta) + R\cot(s\alpha)\sin(s\theta)], \tag{30}$$

$$\lim_{r\to 0} \tau_{\theta z} = \lim_{s\to s_1} r^{-(s+1)}(s-s_1)\hat{\tau}_{\theta z}(s,\theta) = r^{-\lambda} \lim_{s\to s_1} \frac{B(s)}{D'(s)}[\sin(s\theta) - R\cot(\alpha s)\cos(s\theta)], \tag{31}$$

$$\lim_{r\to 0} \tau^*_{rz} = \lim_{s\to s_1} r^{-(s+1)}(s-s_1)\hat{\tau}^*_{rz}(s,\theta) = r^{-\lambda} \lim_{s\to s_1} \frac{B(s)}{D'(s)} R[\cos(s\theta) + \cot(\alpha s)\sin(s\theta)], \tag{32}$$

$$\lim_{r\to 0} \tau^*_{\theta z} = \lim_{s\to s_1} r^{-(s+1)}(s-s_1)\hat{\tau}^*_{\theta z}(s,\theta) = r^{-\lambda} \lim_{s\to s_1} \frac{B(s)}{D'(s)} R[\sin(s\theta) - \cot(\alpha s)\cos(s\theta)], \tag{33}$$

where

$$B(s) = -\mu[s\mu^* \hat{W}^* \cos(s\beta) + \hat{t}(s)\sin(s\alpha)].$$

The stress component $\tau_{\theta z}$ is continuous at $\theta = 0$ ($\tau^*_{\theta z}(r,0^-) = \tau_{\theta z}(r,0^+)$) while τ_{rz} is discontinuous there. From (30) and (32), the jump value of τ_{rz} will be

$$\lim_{r\to 0}[\tau_{rz}(r,0^+) - \tau^*_{rz}(r,0^-)] = (1-R)r^{-(s_1+1)} \frac{B(s_1)}{D'(s_1)}. \tag{34}$$

The angular dependence of stresses near the wedge apex for $\alpha = \pi$, $\beta = \pi/2$ and $R = 0.5$, 2.0 are shown in Figs 9 and 10.

Case III. Displacement–displacement boundary condition

We consider displacements prescribed at the boundary faces $\theta = -\alpha$ and $\theta = \beta$ of the form,

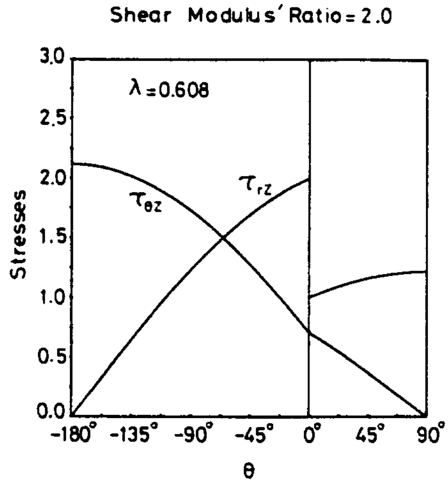

Fig. 10. The angular distribution of stresses $\tau_{\theta z}$ and τ_{rz} of the asymptotic behavior as $r \to 0$ for traction-displacement boundary condition ($\alpha = \pi$, $\beta = \pi/2$, $R = 2.0$).

$$w^*(r, -\alpha) = W^*(r),$$
$$w(r, \beta) = W(r). \tag{35}$$

The stress solution in the transformed form will be

$$\hat{\tau}_{rz}(s, \theta) = -\frac{s\mu}{D}\{\mu^*[\hat{W} \cos(s\alpha) - \hat{W}^* \cos(s\beta)] \sin(s\theta)$$
$$+[\mu\hat{W} \sin(s\alpha) + \mu^*\hat{W}^* \sin(s\beta)] \cos(s\theta)\}$$
$$\hat{\tau}_{\theta z}(s, \theta) = \frac{s\mu}{D}\{\mu^*[\hat{W} \cos(s\alpha) - \hat{W}^* \cos(s\beta)] \cos(s\theta)$$
$$-[\mu\hat{W} \sin(s\alpha) + \mu^*\hat{W}^* \sin(s\beta)]\} \tag{36}$$

where

$$D(s) = \mu^* \sin(s\beta) \cos(s\alpha) + \mu \cos(s\beta) \sin(s\alpha), \tag{37}$$

then $D(s) = 0$ yields

$$\frac{\sin[(\alpha+\beta)s]}{\sin[(\alpha-\beta)s]} = \frac{1-1/R}{1+1/R}. \tag{38}$$

Equation (38) has exactly the same form as (23) except replacing R with $1/R$. The asymptotic stress field is very similar to that in (30)–(33) except that $B(s)$ is replaced by $C(s) = -s\mu[\mu\hat{W} \sin(s\alpha) + \mu^*\hat{W}^* \sin(s\beta)]$ and $D'(s) = (\mu^*\beta + \mu\alpha) \cos(s\alpha) \cos(s\beta) - (\mu^*\alpha + \mu\beta) \sin(s\alpha) \sin(s\beta)$.

4. STRESS SINGULARITIES AT ANISOTROPIC BIMATERIAL WEDGE

In this section, the problem for two dissimilar anisotropic wedges of arbitrary angle is formulated. The method employs the complex representation of the antiplane anisotropic elasticity solution in conjunction with a generalization of the Mellin transform. This method has been used by Bogy (1972) and Kuo and Bogy (1974a, b) on the in-plane problems. Attention is also focused on the dependence of the order of the power singularities in the stress field at the apex on the wedge angle and material constants. If the plane of elastic symmetry is assumed to be normal to the z axis, then there are only three relevant coefficients

c_{44}, c_{45} and c_{55} to be considered. The stress components are related to the displacement as follows

$$\tau_{yz} = c_{44}\frac{\partial w}{\partial y} + c_{45}\frac{\partial w}{\partial x}, \tag{39}$$

$$\tau_{xz} = c_{45}\frac{\partial w}{\partial y} + c_{55}\frac{\partial w}{\partial x}. \tag{40}$$

The corresponding displacement equation of equilibrium is

$$c_{55}\frac{\partial^2 w}{\partial x^2} + 2c_{45}\frac{\partial^2 w}{\partial x \partial y} + c_{44}\frac{\partial^2 w}{\partial y^2} = 0. \tag{41}$$

The governing equation (41) can be solved in the complex plane $z = x + py$ such that

$$w(x, y) = 2\ \mathrm{Re}\ [U(z)], \tag{42}$$

where U is an arbitrary function of z and p is a value dependent on the elasticity constants. Substitution of (42) into (41) yields p must satisfy the characteristic equation

$$c_{44}p^2 + 2c_{45}p + c_{55} = 0. \tag{43}$$

Hence

$$p = \frac{-c_{45} \pm \mathrm{i}\sqrt{c_{44}c_{55} - (c_{45})^2}}{c_{44}}.$$

It is expedient to define

$$\phi(z) = \mathrm{i}\sqrt{c_{44}c_{55} - (c_{45})^2}\,\frac{\mathrm{d}U}{\mathrm{d}z}, \tag{44}$$

so that the shear stresses may be written simply as

$$\tau_{xz} = -(p\phi + \bar{p}\bar{\phi}), \tag{45}$$

$$\tau_{yz} = \phi + \bar{\phi}, \tag{46}$$

where the overline denotes the complex conjugate. Consider the stress transformation

$$\tau_{\theta z} = \tau_{yz}\cos\theta - \tau_{xz}\sin\theta, \tag{47}$$

$$\tau_{rz} = \tau_{yz}\sin\theta + \tau_{xz}\cos\theta. \tag{48}$$

The solution of the problem is obtained by use of an integral transform which is a complex analogy to the standard Mellin transform. Following a procedure similar to Bogy (1972), let $\hat{U}(s)$ be defined by

$$\hat{U}(s) = \int_0^\infty U(z)z^{s-1}\,\mathrm{d}z = (\cos\theta + p\sin\theta)^s \int_0^\infty U(z)r^{s-1}\,\mathrm{d}r \tag{49}$$

in which the path of integration is along a ray of fixed θ and s is a complex transform parameter. We obtain also from the conjugate of (49)

$$\hat{\bar{U}}(s) = \int_0^\infty \bar{U}(\bar{z})\bar{z}^{s-1}\, d\bar{z} = (\cos\theta+\bar{p}\sin\theta)^s \int_0^\infty \bar{U}(\bar{z})r^{s-1}\, dr. \tag{50}$$

From a formal integration by parts and with appropriately assumed behavior as $r \to 0$ and ∞, we have

$$\int_0^\infty U'(z)r^s\, dr = -\frac{s\hat{U}(s)}{(\cos\theta+p\sin\theta)^{s+1}}. \tag{51}$$

$$\int_0^\infty \bar{U}'(\bar{z})r^s\, dr = -\frac{s\hat{\bar{U}}(s)}{(\cos\theta+\bar{p}\sin\theta)^{s+1}}. \tag{52}$$

If the integral operation is applied to (42) and (47) and use is made of (49)–(52) there follows

$$\hat{\tau}_{\theta z}(s,\theta) = -\mathrm{i}C\left[\frac{s\hat{U}(s)}{H(\theta)} - \frac{s\hat{\bar{U}}(s)}{\bar{H}(\theta)}\right], \tag{53}$$

$$\hat{w}(s,\theta) = \frac{\hat{U}(s)}{H(\theta)} + \frac{\hat{\bar{U}}(s)}{\bar{H}(\theta)}, \tag{54}$$

in which

$$C = [c_{44}c_{55}-(c_{45})^2]^{1/2}, \quad H(\theta) = (\cos\theta+p\sin\theta)^s.$$

In the same definition as the isotropic case, $\hat{\tau}_{\theta z}(s,\theta)$ is the Mellin transform with respect to r of $r\tau_{\theta z}(r,\theta)$. The traction prescribed boundary conditions as shown in (16) in conjunction with (53), (54) yield for the determination of the four unknowns $\hat{U}(s)$, $\hat{U}^*(s)$ etc., for the following inhomogeneous system of four equations,

$$\begin{aligned} C^*\hat{U}^* - C^*\hat{\bar{U}}^* - C\hat{U} + C\hat{\bar{U}} &= 0, \\ \hat{U}^* + \hat{\bar{U}}^* - \hat{U} - \hat{\bar{U}} &= 0, \\ \frac{\hat{U}^*}{H(-\alpha)} - \frac{\hat{\bar{U}}^*}{\bar{H}(-\alpha)} &= \frac{\hat{t}^*(s)}{-\mathrm{i}C^*s}, \\ \frac{\hat{U}}{H(\beta)} - \frac{\hat{\bar{U}}}{\bar{H}(\beta)} &= \frac{\hat{t}(s)}{-\mathrm{i}Cs}. \end{aligned} \tag{55}$$

This system can be easily solved and the expressions for $\hat{\tau}_{\theta z}(s,\theta)$, $\hat{w}(s,\theta)$ now follow directly from substitution of (55) into (53) and (54). This completes the formal solution for the transforms of the stress and displacement components. As discussed in the isotropic material case in the previous section, the dependence of the order of the stress field singularity on the wedge angle and the material parameters is determined by the pole of the meromorphic function $\hat{\tau}_{\theta z}(s,\theta)$ etc., or the location of the zero of the following equation

$$(Q-1)\sin[(\xi-\eta)s]+(Q+1)\sin[(\xi+\eta)s] = 0,$$

or

$$\frac{\sin[(\xi+\eta)s]}{\sin[(\xi-\eta)s]} = \frac{1-Q}{1+Q}, \tag{56}$$

where

$$Q = \frac{\sqrt{c_{44}^* c_{55}^* - (c_{45}^*)^2}}{\sqrt{c_{44} c_{55} - (c_{45})^2}}, \tag{57}$$

$$\tan \xi = \frac{\sqrt{c_{44}^* c_{55}^* - (c_{45}^*)^2} \sin \alpha}{c_{44}^* \cos \alpha + c_{45}^* \sin \alpha}, \tag{58}$$

$$\tan \eta = \frac{\sqrt{c_{44} c_{55} - (c_{45})^2} \sin \beta}{c_{44} \cos \beta - c_{45} \sin \beta}. \tag{59}$$

It is surprising that (56) has exactly the same functional form as (23) for the isotropic case. Here ξ and η are called the effective wedge angle that are defined in (58) and (59). Q is the ratio of material constants of two wedges defined in (57). For the isotropic case, $c_{45} = 0$ and $c_{44} = c_{55} = \mu$, we have $Q = \mu^*/\mu = R$, $\xi = \alpha$ and $\eta = \beta$, then (56) reduces to the isotropic solution as shown in (23). For the interfacial crack problem, $\alpha = \beta = \pi$, we have an effective wedge angle $\xi = \eta = \pi$, so that an interfacial crack in a general anisotropic material in the antiplane problem also gives rise to the square root singularity. As the order of singularity in the present case shares the same feature as that in the isotropic case, the discussion will not be repeated here. However, it is worth mentioning again that the order of the singularity for the anisotropic bimaterial wedge is real for all cases, and that oscillatory singular behavior is not presented.

The problem of traction prescribed on one face with displacement prescribed on the other as shown in the boundary condition (24) can be analyzed in a similar way. The result is

$$\frac{\cos [(\xi+\eta)s]}{\cos [(\xi-\eta)s]} = \frac{1-Q}{1+Q}. \tag{60}$$

Again, (60) has exactly the same functional form as (28) for Q, ξ and η defined in (57)–(59). For the special case of the interfacial crack problem, $\alpha = \beta = \pi$, the order of the stress singularity can be expressed as follows,

$$\lambda = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \left[\frac{\sqrt{c_{44} c_{55} - (c_{45})^2}}{\sqrt{c_{44}^* c_{55}^* - (c_{45}^*)^2}} \right]. \tag{61}$$

Finally, for prescribed displacements at both boundary faces as indicated in (35), the order of the stress singularity is obtained from solving the following equation

$$\frac{\sin [(\xi+\eta)s]}{\sin [(\xi-\eta)s]} = \frac{1-1/Q}{1+1/Q}. \tag{62}$$

5. CONCLUDING REMARKS

The problem of antiplane shear for dissimilar anisotropic bimaterial wedges was solved by a straightforward application of the Mellin transform. Emphasis is placed on the investigation of the order of the singularity and the angular dependence in the stress field at the apex. It is shown in this paper that the order of the stress singularity λ is always real for the antiplane anisotropic dissimilar wedge. This is a quite different matter from the in-plane case, in which λ may be complex. If an effective angle is introduced in the analysis for the anisotropic case, then the characteristic equation which determines the order of the stress singularity, has the same functional form as that for the isotropic case. These results may simplify analysis of the antiplane anisotropic wedge problem.

Explicit solutions of the order of stress singularity were obtained for some special cases, viz. equal angle wedge and interfacial crack problems. It has been shown that the familiar square root singularity is obtained for dissimilar anisotropic materials with a crack having traction–traction and displacement–displacement boundary conditions. While for the traction–displacement boundary condition, the order of stress singularity of dissimilar anisotropic materials with cracks will depend on the material constants. For the anisotropic crack problem of one material only, the order of stress singularity will be 3/4, the same as for the isotropic case.

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REFERENCES

Benthem, J. P. (1963). On the stress distribution in anisotropic wedges. *Q. Appl. Math.* **21**, 189–198.

Bogy, D. B. (1971). Two edge-bonded elastic wedges of different materials and wedge angles under surface tractions. *J. Appl. Mech.* **38**, 377–386.

Bogy, D. B. (1972). The plane solution for anisotropic elastic wedges under normal and shear loading. *J. Appl. Mech.* **39**, 1103–1109.

Clements, D. L. (1971). A crack between dissimilar anisotropic media. *Int. J. Engng Sci.* **9**, 257–265.

Delale, F. and Erdogan, F. (1979). Bonded orthotropic strips with cracks. *Int. J. Fracture* **15**, 343–364.

Dempsey, J. P. and Sinclair, G. B. (1981). On the singular behavior at the vertex of a bimaterial wedge. *J. Elasticity* **11**, 317–327.

Erdogan, F. and Gupta, G. D. (1972). Stresses near a flat inclusion in bonded dissimilar materials. *Int. J. Solids Structures* **8**, 533–547.

Green, A. E. and Zerna, W. (1954). *Theoretical Elasticity*. Clarendon, Oxford.

Hoenig, A. (1982). Near-tip behavior of a crack in a plane anisotropic elastic body. *Engng Fract. Mech.* **16**, 393–403.

Kuo, M. C. and Bogy, D. B. (1974a). Plane solutions for the displacement and traction–displacement problems for anisotropic elastic wedges. *J. Appl. Mech.* **41**, 197–202.

Kuo, M. C. and Bogy, D. B. (1974b). Plane solutions for traction problems on orthotropic unsymmetrical wedges and symmetrically twinned wedges. *J. Appl. Mech.* **41**, 203–208.

Sih, G. C., Paris, P. C. and Irwin, G. R. (1965). On cracks in rectilinearly anisotropic bodies. *Int. J. Fract. Mech.* **1**, 189–302.

Stroh, A. N. (1958). Dislocations and cracks in anisotropic elasticity. *Phil. Mag.* **7**, 625–646.

Stroh, A. N. (1962). Steady state problems in anisotropic elasticity. *J. Math. Phys.* **41**, 77–103.

Ting, T. C. T. (1986). Explicit solution and invariance of the singularities at an interface crack in anisotropic composites. *Int. J. Solids Structures* **22**, 965–983.

Ting, T. C. T. and Chou, S. C. (1981). Edge singularities in anisotropic composites. *Int. J. Solids Structures* **17**, 1057–1068.

Tranter, C. J. (1948). The use of the Mellin transform in finding the stress distribution in an infinite wedge. *Q. J. Mech. Appl. Math.* **1**, 125–130.

Wang, S. S. and Choi, I. (1982a). Boundary-layer effects in composite laminates: Part 1—Free-edge stress singularities. *J. Appl. Mech.* **49**, 541–548.

Wang, S. S. and Choi, I. (1982b). Boundary-layer effects in composite laminates: Part 2—Free-edge stress solutions and basic characteristics. *J. Appl. Mech.* **49**, 549–560.

Williams, M. L. (1952). Stress singularities resulting from various boundary conditions in angular corners of plates in extension. *J. Appl. Mech.* **19**, 526–528.

Williams, M. L. (1959). The stresses around a fault or crack in dissimilar media. *Bull. Seism. Soc. Am.* **49**, 199–204.

APPENDIX

From eqn (23)

$$\frac{\sin[(\alpha+\beta)s]}{\sin[(\alpha-\beta)s]} = \frac{1-R}{1+R}. \tag{A1}$$

We have

$$\sin[(\alpha+\beta)s] = \Omega \sin[(\alpha-\beta)s], \tag{A2}$$

where

$$-1 \leqslant \Omega = \frac{1-R}{1+R} \leqslant 1.$$

Assume (A2) has a complex root of the form $s = x+\mathrm{i}y$ and $x \neq 0$, $y \neq 0$, then (A2) can be rewritten as follows

$$\sin[(\alpha+\beta)x]\cosh[(\alpha+\beta)y] + \mathrm{i}\cos[(\alpha+\beta)x]\sinh[(\alpha+\beta)y]$$
$$= \Omega\sin[(\alpha-\beta)x]\cosh[(\alpha-\beta)y] + \mathrm{i}\Omega\cos[(\alpha-\beta)x]\sinh[(\alpha-\beta)y]. \tag{A3}$$

Equating the real and imaginary parts of (A3) yields

$$\sin[(\alpha+\beta)x]\cosh[(\alpha+\beta)y] = \Omega\sin[(\alpha-\beta)x]\cosh[(\alpha-\beta)y], \tag{A4}$$

$$\cos[(\alpha+\beta)x]\sinh[(\alpha+\beta)y] = \Omega\cos[(\alpha-\beta)x]\sinh[(\alpha-\beta)y]. \tag{A5}$$

Then (A4) and (A5) can be combined into the following equation

$$\sin^2[(\alpha+\beta)x]\left\{\frac{\cosh[(\alpha+\beta)y]}{\cosh[(\alpha-\beta)y]}\right\}^2+\cos^2[(\alpha+\beta)x]\left\{\frac{\sinh[(\alpha+\beta)y]}{\sinh[(\alpha-\beta)y]}\right\}^2 = \Omega^2. \tag{A6}$$

Since $|(\alpha+\beta)y| > |(\alpha-\beta)y|$, hence

$$\left\{\frac{\cosh[(\alpha+\beta)y]}{\cosh[(\alpha-\beta)y]}\right\}^2 > 1,$$

and

$$\left\{\frac{\sinh[(\alpha+\beta)y]}{\sinh[(\alpha-\beta)y]}\right\}^2 > 1,$$

which makes the left hand side of (A6) greater than 1. But the right hand side of (A6) is always less than 1 and we have a contradiction. If $x = 0$ and $y \neq 0$, from (A3) we also get a contradictory result. Hence the only possibility of finding the solution of (A2) is for $x \neq 0$, $y = 0$ which indicates that the order of stress singularity is real and this completes the proof.